READING SEMINAR 2025 FALL

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1. Langlands-Kottwitz methods on modular curve

The Langlands program is a central topic in mathematics, relating algebraic geometry, number theory, and representation theory. It predicts deep relationships between automorphic representations and Galois representations. Let G be an reductive group over a global field F and let $\widehat{G}$ be its dual group. Roughly speaking, the Langlands correspondence is searching for the relation between

$$\left\{\text{Automorphic representation of } G(\mathbb{A}_F)\right\} \longleftrightarrow \left\{\text{Galois representation } \Gamma_F \to \widehat{G}\right\}$$

The bijection are excepted to satisfy certain nice properties, for example, we except the L-function constructed in two sides (which are in a totally different flavor) agrees.

Shimura varieties are certain algebraic varieties defined by group-theoretic data. Their cohomology carry both Hecke actions and Galois actions, which allows one to prove important instances of the Langlands correspondence. The Langlands-Kottwitz method [Lan77, Lan79b, Lan79a, Kot92] aims to study the cohomology of Shimura varieties via their integral models and their reduction mod p. The aim of this reading seminar is to realize this idea over the modular curves, following the master thesis of Peter Scholze [Sch11].

Let X be a variety over a number field K. Recall that the Hasse-Weil zeta function of X is defined as the Euler product of local factors:

$$\zeta(X,s) := \prod_{v \text{ finite places of } K} \zeta_v(X_v, s),$$

where, for each finite place v, the local factor is given by

$$\zeta(X_v,s) := \prod_{i=0}^{2\dim X} \det(1 - \Phi_q q^{-s} | H_c^i(X \otimes_{K_v} \overline{K}_v, \overline{\mathbb{Q}}_\ell)^{I_{K_v}})^{(-1)^i}.$$

Here, $I_{K_v} \subset \mathrm{Gal}(\overline{K}_v, K_v)$ denotes the inertia subgroup and Φ_q is a lift of the Frobenius element from the residue field $\mathbb{F}_q$. The Hasse–Weil zeta function encodes important information about the associated Galois representations.

Let $\mathcal{M}_m/\mathbb{Z}[1/m]$ be the moduli space of modular curve with level-m-structure and let $\overline{\mathcal{M}}_m$ be its compactification. The ultimate goal is to prove the following identity:

$$\zeta(\overline{\mathcal{M}}_{m,\mathbb{Q}}, s) = \prod_{\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)} L\Big(\pi, s - \frac{1}{2}\Big)^{\frac{1}{2} m(\pi)\chi(\pi_\infty)\dim \pi_f^{K_m}}, \tag{1.1}$$

where:

- $K_m = \{g \in \mathrm{GL}(\widehat{\mathbb{Z}}) | \, g \equiv 1 \mod m\}$ is the principal congruence subgroup of level m.
- $\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}), 1)$ is the set of automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $\mathrm{GL}_2(\mathbb{A})$ that occurs discretely in $L^2(\mathrm{GL}_2(\mathbb{Q})\mathbb{R}^\times \backslash \mathrm{GL}_2(\mathbb{A}))$, such that π_∞ has trivial central and infinitesimal character.
- $m(\pi)$ is the multiplicity of π in the discrete spectrum, which equals 1 by the multiplicity-one theorem.
- $\chi(\pi_\infty)$ is 2 if π_∞ is a character and -2 otherwise.

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The rough idea toward the formula (1.1) is as follows:
Step 1. Suppose $\overline{\mathcal{M}}_m$ is smooth over $\mathbb{Z}_{(p)}$. Then, by the proper base change and the Lefschetz trace formula, the Hasse-Weil local factor at the place p can be computed using point counting formula:

$$\log \zeta(\overline{\mathcal{M}}_{m,\mathbb{Q}_p}, s) = \sum_{r \geq 0} |\overline{\mathcal{M}}_m(\mathbb{F}_{p^r})| \frac{p^{-rs}}{r},$$

where $|\overline{\mathcal{M}}_m(\mathbb{F}_{p^r})|$ counts the $\mathbb{F}_{p^r}$-points of the special fiber of the modular curve.

However, several issues arise in general:

(1) The moduli integration of $\overline{\mathcal{M}}_m$ is only defined over $\mathbb{Z}[1/m]$. To extend it to a well-behaved integral model at $p \mid m$, one must carefully revisit the moduli interpretation of modular curves. This involves the notion of Drinfeld level structures, as developed in [KM85].

(2) Even when an integral model is availble over $\mathbb{Z}_{(p)}$, it is generally not smooth. To relate the Hasse-Weil zeta function over the generic fiber to point counts over the special fiber, one must go beyond the proper base change theorem and instead compute nearby cycles, incorporating semisimple L-functions. This requires a deeper analysis of the geometry of the integral model and the use of more advanced tools from algebraic geometry.

(3) The modular curve are not compact, so one must consider their compactifications and compute the corresponding local factors there.

Step 2. Using the moduli interpretation of modular curves, the point counting for $|\overline{\mathcal{M}}_m(\mathbb{F}_{p^r})|$ reduces to counting elliptic curves with additional structure over finite fields up to quasi-isogeny. By applying the theories of Tate modules and Dieudonné modules, this point-counting problem can be further translated into a lattice-counting problems over $\mathbb{A}_f$. This is analogous to the complex uniformization of elliptic curves over $\mathbb{C}$, where elliptic curves correspond to rank two lattices in the complex plane. Similarly, in the p-adic setting, the classification of elliptic curves can be interpreted via certain lattice data.

The resulting lattice counting problem then can be expressed in terms of orbital integrals. A technical subtlety arises at the prime p: morphism between Dieudonné modules are defined only up to Frobenius twists, due to the semi-linear nature of Dieudonné theory. As a result, the relevant orbital integrals at p are twisted orbital integrals. To relate them to standard orbital integrals, one must invoke the base change fundamental lemma.
Step 3. Having identified the Hasse-Weil zeta function with a sum of orbital integrals, we now apply the Arthur-Selberg trace formula to express this in terms of traces on spaces of automorphic representations. The final step is to compute the trace of the relevant test function and extract the spectral side of the formula.

A technical remark: since we are working with the "semisimple" version of the Hasse-Weil zeta function, a more refined analysis of the cohomology of modular curves is required to recover the full genuine Hasse-Weil zeta function.

2. Talks

Below is a rough plan for the seminar, can be more explicit and provide references later.

2.1. **Talk : Elliptic curves.**

Goal: review basic properties of elliptic curves.

Content: isogeny, Tate module, elliptic curve over DVR.

Reference:

2.2. **Talk : Finite group scheme and Dieudonné module.**

Goal: discuss Dieudonne theory.

Content: basic result on the finite group scheme and Dieudonne theory.

Reference:

2.3. Talk : Elliptic curves over finite field.

Goal: discuss elliptic curve over finite field.

Content: isogeny, p-torsions and Dieudonne theory of elliptic curve. Endomorphism of elliptic curves. Honda–Tate theory ([Sch11, Theorem 10.4]).

Reference:

2.4. Talk : Modular curves.

Goal: cover [Sch11, §4].

Content: modular curve with principal level at good reduction place. If time permit, talk about compactifying modular curves via generalized elliptic curves.

Reference: [DR73].

2.5. Talk : Étale cohomology and crystalline cohomology.

Goal: overview of étale cohomology and crystalline cohomology.

Content: definition of the étale cohomology. Include proper base change and Lefschetz trace formula. Relating H^1 with étale fundamental group and compute the it for elliptic curves. Constructible sheaf. Survey the weight-monodromy conjecture. Give a quick overview of the crystalline cohomology. Focus on the case of elliptic curve and its relation to the Dieudonné theory.

Reference: [DR73].

2.6. Talk : Counting point: good reduction.

Goal: cover [Sch11, §5].

Content: discuss the point counting for elliptic curves at good reduction place and find the local zeta function (without compactification). This involves how to relate elliptic curve over finite field to some lattices in the adeles. Define orbital integral and twisted orbital integral, relate the lattice count with orbital integrals.

Reference: [Sch11, §5].

2.7. Talk : Drinfeld level structure.

Goal: cover [Sch11, §6].

Content: discuss the Drinfeld level structure, define the integral model of modular curve at bad reduction place. If time permit, discuss its local structure and compactification.

Reference: [KM85] and [Hua16].

2.8. Talk : Nearby cycles of étale cohomology.

Goal: cover the discussions in [Sch11, §8] before [Sch11, Corollary 8.5].

Content: discuss derived category of constructible sheaf. Discuss six functor and nearby cycles of étale cohomology and illustrate its geometric intuition. Calculate the nearby cycles for some special cases.

Reference: [RZ82, §2].

2.9. Talk : The semisimple local zeta factor.

Goal: cover [Sch11, §7].

Content: define and study the semisimple local zeta factor.

Reference: [HN02, §3] and [Rap90].

2.10. Talk : Counting point: bad reduction.

Goal: complete [Sch11, §8]. Prove [Sch11, Corollary 9.6] assuming [Sch11, Theorem 9.3]. Prove [Sch11, Corollary 9.8]. Conclude [Sch11, Corollary 10.1].

Content:

Reference:

2.11. **Talk : Representation of p-adic group.**

Goal: overview of representation of p-adic group.

Content: Introduce smooth and admissible representations for p-adic groups (or simply $\mathrm{GL}_2(F)$), Hecke algebra and relation to representation, parabolic induction, parahoric induction, Jacquet modules and supercuspidal representations.

Reference: [Bum97, Chapter 4], [GH11, Chapter 6 and 8], [Fin23], [Taï19].

2.12. **Talk : Trace and character theory of local representation.**

Goal: overview of trace and character theory of local representation.

Content: Trace and character of a smooth representation. Weyl's integration formula, and Harish-Chandra's representability theorem.

Reference: [JL70, §7, §15], [Kal25], [Taï19, §3].

2.13. **Talk : Bernstein decomposition.**

Goal: cover [Sch11, §2].

Content: Bernstein center and Bernstein decomposition.

Reference: [Ber84], [Zha], [Roc09], [Cha16].

2.14. **Talk : Automorphic representation of GL_2.**

Goal: overview of automorphic representation of GL_2.

Content: Automorphic representation for GL_2. Admissible representation, discrete series. Spectral decomposition. Some representation theory at archimedean place.

Reference: [Bum97, §3]

2.15. **Talk : Langlands correspondence, functoriality, and base change.**

Goal: survey Langlands correspondence and functoriality. Motivate the base change problem.

Content: Satake isomorphism and Satake correspondence (at least for GL_2). (Local and global) Langlands correspondence, Langlands functoriality and base change lift.

Reference: [Bum97, §3.9], [Art05, §26].

2.16. **Talk : Arthur-Selberg trace formula.**

Goal: Cover [Sch11, §12].

Content: Introduce trace formula in the general setting, including the basic terminologies. Then explain how it applies to our case and get [Sch11, Theorem 12.1].

Reference: [Art89], [Gel96, Lecture 1], [Sch11, §12], [Art05].

2.17. **Talk : Base change fundamental lemma.**

Goal: cover [Sch11, §3].

Content: Explain and proof the base change fundamental lemma for GL_2. If time permits, explain why it is called the base change fundamental lemma (prove the base change using trace formula).

Reference: [Sch11, §3], [Zha].

2.18. **Talk : Semisimple trace of Frobenius as a twisted orbital integral.**

Goal: Complete [Sch11, §9].

Content: Complete [Sch11, §9]

Reference: Complete [Sch11, §9]

2.19. **Talk : Langlands-Kottwitz approach.**

Goal: complete [Sch11, §10].

Content: Review of computations in good reduction and bad reduction. Apply base change fundamental lemma, and use Honda-Tate theory to further simplify the formula.

Reference: [Sch11, §10].

2.20. **Talk : Langlands-Kottwitz approach: boundary.**

Goal: cover [Sch11, §11].

Content: compactification of modular curve. Computation of boundary. Point-counting on the boundary.

Reference: [Sch11, §11].

2.21. **Talk : Comparison of the Lefschetz and Arthur–Selberg Trace Formula.**

Goal: Cover [Sch11, §13].

Content: Finish [Sch11, §13]

Reference: [Sch11, §13].

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